Problem 1

Let $y = X(t)T(t)$ where $X(x)$ depends only on x
$T(t)$ depends only on T.

Substiting into the differential equation yields

3 points

$$XT'' = 16X''T \quad \text{or} \quad \frac{X''}{X} = \frac{T''}{16T}$$

Set this to be a constant, say $-\lambda^2$

$$X'' + \lambda^2 X = 0$$

2 point

$$T'' + 16\lambda^2 T = 0.$$

Solving we find that

2 points

$$X(x) = a_1 \cos \lambda x + b_1 \sin \lambda x$$
$$T(t) = a_2 \cos 4\lambda t + b_2 \sin 4\lambda t$$

The solution is therefore

$$y(x,t) = (a_1 \cos \lambda x + b_1 \sin \lambda x)(a_2 \cos 4\lambda t + b_2 \sin 4\lambda t)$$

2 points.

We use the bonday conditions

(1) $y(0,t)=0$

(2) $y_t(x,0)=0$

(1) $\Rightarrow a_1(a_2 \cos 4\lambda t + b_2 \sin 4\lambda t) = 0$

This implies $a_1 = 0$ 2 points

(2) $\Rightarrow y_t(x,0) = (b_1 \sin \lambda x)(4\lambda b_2) = 0$

this implies $b_2 = 0$ 2 points

Therefore $y(x,t) = B \sin \lambda x \cos 4\lambda t$.

for $B = b_1 a_2$. 1 point to final sol.

Now from $y(2,t)=0$ we find

2 points

$B \sin 2\lambda \cos 4\lambda t = 0$.

Therefore we must have $\sin 2\lambda = 0$

In otherwords $2\lambda = m\pi$ or

$\lambda = \frac{m\pi}{2} \quad m \in \mathbb{Z}$.

2 points

Therefore $y(x,t) = B \sin \frac{m\pi x}{2} \cos 2m\pi t$.

is a solution. Because this Solution is bounded $|y(x,t)| < M$ is automatically satisfied. 2 points

To satisfy the last condition

$y(x,0) = 5 \sin \pi x - 3 \sin 4\pi x$

we use the principle of superposition to obtain.

$$y(x,t) = B_1 \sin \frac{m_1 \pi x}{2} \cos 2\pi m_1 t$$ 3 points

$$+ B_2 \sin \frac{m_2 \pi x}{2} \cos 2m_2 \pi t.$$

Setting $t=0$ we obtain

$$y(x,0) = 6 \sin \pi x - 3 \sin 4\pi x$$

$\Rightarrow B_1 = 6 \quad m_1 = 2$

$B_2 = -3 \quad m_2 = 8.$

2 points to final answer

Therefore the desired solution is

$$y(x,t) = 6 \sin \pi x \cos 4\pi t$$
$$- 3 \sin 4\pi x \cos 16\pi t .$$

Problem 2

2 points for correct true/false
3 points each for reasoning

a) False. By definition of uniform convergence $\forall \varepsilon\ \exists N$ s.t. $|f_n(x) - \sigma(x)| < \varepsilon \quad \forall n \geq N$. But if $\varepsilon < 1/2$ there is no such N since each $f_n(x)$ would need to be s.t. $f_n(x) < \varepsilon < 1/2$ for $x < 0$ and $f_n(x) > 1 - \varepsilon > 1/2$ for $x > 0$ a cont. to continuity.

b) True

The jumps of $|f(x)|$ are the same as the jumps of $f(x)$ unless $f(x_+) = -f(x_-) \neq 0$ in which case $x = a$ removable discontinuity of $|f(x)|$. (The jump magnitude of $|f(x)|$ at a discontinuity is possible to compute).

c) False ~~True~~. This is theorem 3.8 in the text. The function to which f converges need not be piecewise continuous

Problem 3

A Green's function for the given problem is a function $G(\underline{x},\underline{\xi})$, $\underline{x},\underline{\xi} \in \Omega$ which is

1 point per property = 4 points

- continuous for $\underline{x} \in \Omega \setminus \{\underline{\xi}\}$
- is harmonic for $\underline{x} \in \Omega \setminus \{\underline{\xi}\}$
- Satisfies $G(\underline{x},\underline{\xi}) = 0$ $\underline{x} \in \partial\Omega$
- Satisfies $-\Delta_{\underline{x}} G = \delta(\underline{x}-\underline{\xi})$

To prove G is unique assume we had 2 functions G_1, G_2 with the above properties

Let $G = G_1 - G_2$ so that

$$\left.\begin{aligned} &\Delta G = 0 \text{ in } \Omega \\ &G = 0 \text{ on } \partial\Omega \end{aligned}\right\} \text{ 2 points}$$

Then G is harmonic <u>everywhere</u> in Ω so by the maximum principle $G \equiv 0$.

3 points, must mention max. principle

To prove G is symmetric

$$G(x_1, x_2) = G(x_2, x_1)$$

We apply Green's second identity to a pair of functions

$$v_1(x) = G(x, x_1)$$
$$v_2(x) = G(x, x_2)$$

We find that

$$\left.\begin{aligned} &\int_{\Omega} (v_1 \Delta_x v_2 - v_2 \Delta_x v_1)\, dx = \\ &\int_{\partial\Omega} v_1 \frac{\partial v_2}{\partial n} - v_2 \frac{\partial v_1}{\partial n}\, dx = 0 \end{aligned}\right\} \text{2 points}$$

$$= \int_{\Omega} G(x, x_1)\,\delta(x - x_2) - G(x, x_2)\,\delta(x - x_1)\, dx$$

2 points

$$= G(x_2, x_1) - G(x_1, x_2)$$

2 points to this point

1 point to conclusion

Write the boundary value problem as

$$u = f(y) \quad \text{in } \Omega$$
$$u = g(y) \quad \text{on } \partial\Omega$$

The problem for G as

$$\Delta_y G = \delta(x-y) \quad \text{in } \Omega$$
$$G = 0 \quad \text{on } \partial\Omega$$

Then

4 points for all arithmetic which is also in the book.

$$\int_\Omega G\Delta_y u - u\Delta_y G \, dy =$$

$$\int_\Omega f(y) G \, dy - \int_\Omega \delta(x-y) v(y) \, dy$$

$$\Rightarrow \int_{\partial\Omega} \underbrace{\left(G\frac{\partial v}{\partial n_y} - v\frac{\partial G}{\partial n_y}\right)}_{= 0} dy = \int_\Omega f(y) G \, dy - v$$

Then

$$u = \int_\Omega G(x,y) f(y) \, dy + \int_{\partial\Omega} \frac{\partial G}{\partial n_y} g(y) \, dy$$

Part 6

This is a rescaling of Example 6.9 in the text. There $c=1$ and the interval instead is 0 to 1.

$\boxed{\text{5 points total}}$

Then

$$-u'' = \delta(x-\xi)$$

Solution is

$$u'(x) = -\sigma(x-\xi) + a$$

a is a constant.

Integrating again gives

$$u(x) = -\frac{\rho(x-\xi)}{\pi} + ax + b$$

where ρ is a ramp function. Integration constants are fixed by the boundary conditions

$$u(0) = b = 0$$

$$u(\pi) = -\frac{(\pi-\xi)}{\pi} + a + b = 0$$

$$\Rightarrow b = 0 \quad a = \frac{\pi-\xi}{\pi}$$

Therefore

$$G(x,\xi) = \frac{(\pi-\xi)x - (\rho(x-\xi))}{\pi}$$

$$= \begin{cases} \dfrac{(\pi-\xi)\,x}{\pi} & x < \xi \\ \dfrac{\xi\,(\pi-x)}{\pi} & x > \xi \end{cases}$$

Problem 4

The Fourier transform of $u(t,x)$ is

$$\hat{u}(t,k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t,x) e^{-ikx} dx$$ 2 points

Fourier transforming the heat equation gives

$$\frac{\partial \hat{u}}{\partial t} = -k^2 \hat{u}$$ 3 points

$$\hat{f}(k) = \hat{u}(0,k) = \widehat{e^{-x^2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\pi}\, e^{-k^2/4} = \frac{1}{\sqrt{2}} e^{-k^2/4}$$

5 points

The solution to the initial value problem is

$$\hat{u}(t,k) = e^{-k^2 t} \hat{f}(k)$$

$$\hat{u}(t,k) = \frac{1}{\sqrt{2}} e^{-k^2 t} e^{-k^2/4} = \frac{e^{-k^2(t+\frac{1}{4})}}{\sqrt{2}}$$

5 points

Therefore taking the inverse Fourier transform we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{-k^2(t+1/4)}}{\sqrt{2}} e^{ikx}\, dk =$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{t+1/4}}\, \frac{1}{\sqrt{2}}\, e^{-x^2/4(t+1/4)}$$ 5 points

$$= \frac{1}{2\sqrt{(t+1/4)}}\, e^{-\frac{x^2}{4(t+1/4)}}$$

b) as $t \to \infty$ then $e^{-\frac{x^2}{4(t+1/4)}} \to 1$

and $\frac{1}{2\sqrt{t+1/4}} \to 0$ The solution

goes to 0.

5 points for some kind of reasoning as above.